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# The coupled HD hierarchy and a classical integrable system of the complex form

Zhang Baocai and Gu Zhuqan

Shijiazhuang Railway Institute, People's Republic of China

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**Abstract.** In this paper, the coupled Harry Dym (HD) equations are discussed by means of the complex form of the real involutive system. Using the nonlinearization of Lax pairs of the coupled HD equation, a finite-dimensional completely integrable system in the Liouville sense is obtained. By making use of the solutions of commutative flows, the representation of the solutions for the hierarchy of the HD equation are generated.

## 1. Introduction

The Liouville–Arnold theory [1] of the finite-dimensional completely integrable system is beautiful. The relation between the soliton system and the finite-dimensional completely integrable system has been an important topic [2]. Flaschka [3] pointed out an important principle in producing finite-dimensional integrable systems by constraining the infinite-dimensional integrable systems on a finite-dimensional invariant manifold. However, it is not easy to realize. Not long ago Cao Cewen developed a systematic approach [4] to get a finite-dimensional integrable system by the nonlinearization of a Lax pair of soliton equation under certain constraints between potentials and eigenfunctions. But the result of the complete integrability on the complex space is not known. Recently a systematic approach has been given by Gu Zhuqan [5]. The complete integrability of the complex involutive system is proved by means of this approach, and using the nonlinearization [6] of Lax pairs of the soliton equations, many completely integrable systems of the complex form have been obtained. In the present paper, on the real space $\mathbb{R}^{2n}$, the suitable symplectic construction, Poisson bracket and Hamiltonian canonical equation are introduced, therefore the symplectic construction, Poisson bracket and Hamiltonian canonical equation are all written in the complex form. By making use of the nonlinearization of Lax pairs of the coupled Harry Dym (HD) equation [7], a finite-dimensional completely integrable system of the complex form is given. Furthermore, the representation of solutions of the coupled HD heirarchy is generated by using commutable flows of the finite-dimensional completely integrable system.

## 2. Symplectic construction [1]

In order to generate the finite-dimensional completely integrable Hamiltonian system of the complex form, we consider the symplectic construction of the basic coordinate functions $P_1, P_2, \ldots, P_{2N}, Q_1, Q_2, \ldots, Q_{2N}$ in $\mathbb{R}^{4N}$ as follows:

$$\omega=\sum_{j=1}^{2N} \mathrm{d}P_j \wedge \mathrm{d}Q_j. \tag{2.1}$$

Therefore the Poisson bracket of two Hamiltonian functions $H$, $F$ on the symplectic space $(\mathbb{R}^{4N}, \omega=\Sigma_{j=1}^{2N} \mathrm{d}P_j \wedge \mathrm{d}Q_j)$ is defined as

$$(H, F)=\sum_{j=1}^{2N} \frac{\partial H}{\partial Q_j}\frac{\partial F}{\partial P_j}-\frac{\partial H}{\partial P_j}\frac{\partial F}{\partial Q_j}. \tag{2.2}$$

$H$, $F$ is called an involution if $(H, F)=0$.

The Hamiltonian canonical equation of the Hamiltonian function $H$ on $(\mathbb{R}^{4N}, \omega=\Sigma_{j=1}^{2N} \mathrm{d}P_j \wedge \mathrm{d}Q_j)$ is defined as

$$P_{jt}=(P_j, H)=-\frac{\partial H}{\partial Q_j} \qquad Q_{jt}=(Q_j, H)=\frac{\partial H}{\partial P_j} \qquad j=1,2,\ldots,2N. \tag{2.3}$$

*Theorem 2.1.* Let

$$\mathrm{i}=\sqrt{-1} \qquad P_j=\frac{1}{\sqrt{2}}(\varphi_{1j}+\varphi_{1j}^*) \qquad P_{N+j}=\frac{\mathrm{i}}{\sqrt{2}}(\varphi_{2j}-\varphi_{2j}^*)$$

$$Q_j=\frac{1}{\sqrt{2}}(\varphi_{2j}+\varphi_{2j}^*) \qquad Q_{N+j}=\frac{\mathrm{i}}{\sqrt{2}}(\varphi_{1j}-\varphi_{1j}^*)$$

('$*$' denotes complex conjugate), $j=1,2,\ldots,N$. Then the symplectic construction (2.1), Poisson bracket (2.2) and Hamiltonian canonical equation (2.3) are written equivalent as follows in complex form:

$$\sum_{j=1}^{2N} \mathrm{d}P_j \wedge \mathrm{d}Q_j=\sum_{j=1}^{N} \mathrm{d}\varphi_{1j} \wedge \mathrm{d}\varphi_{2j}+\mathrm{d}\varphi_{1j}^* \wedge \mathrm{d}\varphi_{2j}^* \tag{2.4}$$

$$(H, F)=\sum_{j=1}^{N} \frac{\partial H}{\partial \varphi_{2j}}\frac{\partial F}{\partial \varphi_{1j}}-\frac{\partial H}{\partial \varphi_{ij}}\frac{\partial F}{\partial \varphi_{2j}}+\frac{\partial H}{\partial \varphi_{2j}^*}\frac{\partial F}{\partial \varphi_{1j}^*}-\frac{\partial H}{\partial \varphi_{ij}^*}\frac{\partial F}{\partial \varphi_{2j}^*} \tag{2.5}$$

$$\begin{aligned} \varphi_{1jt}&=(\varphi_{ij}, H)=-\frac{\partial H}{\partial \varphi_{2j}} \qquad & \varphi_{1jt}^*&=(\varphi_{1j}^*, H)=-\frac{\partial H}{\partial \varphi_{2j}^*} \\ \varphi_{2jt}&=(\varphi_{2j}, H)=\frac{\partial H}{\partial \varphi_{1j}} \qquad & \varphi_{2jt}^*&=(\varphi_{2j}^*, H)=\frac{\partial H}{\partial \varphi_{1j}^*}. \end{aligned} \tag{2.6}$$

*Proof.* Since

$$\varphi_{1j}=\frac{1}{\sqrt{2}}(P_j-\mathrm{i}Q_{N+j}) \qquad \varphi_{1j}^*=\frac{1}{\sqrt{2}}(P_j+\mathrm{i}Q_{N+j})$$

$$\varphi_{2j}=\frac{1}{\sqrt{2}}(Q_j-\mathrm{i}P_{N+j}) \qquad \varphi_{2j}^*=\frac{1}{\sqrt{2}}(Q_j+\mathrm{i}P_{N+j}).$$

From (2.1), (2.2) and (2.3), by direct computing, (2.4), (2.5) and (2.6) are obtained. Set

$$\begin{aligned} \Phi&=(\Phi_1,\ldots,\Phi_{2N})^T=(\varphi_{11},\ldots,\varphi_{1N},\varphi_{11}^*,\ldots,\varphi_{1N}^*)^T \\ \Psi&=(\Psi_1,\ldots,\Psi_{2N})^T=(\varphi_{21},\ldots,\varphi_{2N},\varphi_{21}^*,\ldots,\varphi_{2N}^*)^T \end{aligned} \tag{2.7}$$

then (2.6), (2.4) and (2.5) are written equally in the complex form as follows:

$$\omega=\sum_{j=1}^{2N} \mathrm{d}P_j\wedge \mathrm{d}Q_j=\sum_{j=1}^{2N} \mathrm{d}\Phi_j\wedge \mathrm{d}\Psi_j \tag{2.8}$$

$$(H,F)=\sum_{j=1}^{2N}\frac{\partial H}{\partial \Psi_j}\frac{\partial F}{\partial \Phi_j}-\frac{\partial H}{\partial \Phi_j}\frac{\partial F}{\partial \Psi_j} \tag{2.9}$$

$$\Phi_{jt}=(\Phi_j,H)=-\frac{\partial H}{\partial \Psi_j}\qquad \Psi_{jt}=(\Psi_j,H)=\frac{\partial H}{\partial \Phi_j}. \tag{2.10}$$

In consideration of the real forms (2.1), (2.2) and (2.3), which are equivalent respectively to the complex forms (2.8), (2.9) and (2.10) (or (2.4), (2.5) and (2.6)), we compute using the complex forms as follows.

We define

$$\langle f,g\rangle=\sum_{j=1}^{n} f_j g_j \tag{2.11}$$

where $f=(f_1,f_2,\ldots,f_n)^T$, $g=(g_1,g_2,\ldots,g_n)^T$.

Let $2N$ complex $\lambda_1,\lambda_2,\ldots,\lambda_N,\lambda_1^*,\ldots,\lambda_N^*$ be different,

$$D=\mathrm{diag}(\lambda_1,\ldots,\lambda_N,\lambda_{N+1},\ldots,\lambda_{2N})$$

where

$$\lambda_{N+j}=\lambda_j^*,\ 1\leqslant j\leqslant N.$$

*Lemma 2.1.* Define

$$\tilde G_k=\sum_{j=1}^{2N}\frac{B_{kj}^2}{\lambda_k-\lambda_j}\qquad B_{kj}=\Phi_k\Psi_j-\Phi_j\Psi_k$$

then:

(i) $(\tilde G_k,\tilde G_j)=0\qquad \forall k,j=1,\ldots,2N.$

(ii) $(\langle\Phi,\Phi\rangle,\tilde G_k)=(\langle\Phi,\Psi\rangle,\tilde G_k)=(\langle\Psi,\Psi\rangle,\tilde G_k)=0.$

(iii) $(\tilde G_k,\Phi_j^2)=4\dfrac{B_{kj}\Phi_k\Phi_j}{\lambda_k-\lambda_j}\qquad (\tilde G_k,\Psi_j^2)=4\dfrac{B_{kj}\Psi_j\Psi_k}{\lambda_k-\lambda_j}.$

(iv) $(\tilde G_k,\Phi_j\Psi_j)=2\dfrac{B_{kj}}{\lambda_k-\lambda_j}(\Phi_k\Psi_j+\Phi_j\Psi_k).$

*Proof.* From (2.9) by direct calculation (or see [8]). □

*Theorem 2.2.* $E_1, E_2,\ldots,E_{2N}$ defined as follows compose an involutive system (i.e. $(E_k,E_j)=0;\ k,j=1,2,\ldots,2N$):

$$E_k=\tfrac12\langle\Psi,\Psi\rangle\Phi_k^2-\langle\Phi,\Psi\rangle\Phi_k\Psi_k+\tfrac12\langle\Phi,\Phi\rangle\Psi_k^2-\tfrac12\langle D\Phi,\Phi\rangle^{-1}\lambda_k\Phi_k^2$$
$$+\tfrac12\langle D\Phi,\Phi\rangle^{-2}\langle D^2\Phi,\Phi\rangle\Phi_k^2-\tfrac12\lambda_k\tilde G_k. \tag{2.12}$$

*Proof.* The theorem is proved by lemma 2.1 via direct calculation. □

*Theorem 2.3.*

(i) The real Hamiltonian function $H_m$ defined as follows is in involution in pairs: $(H_n, H_k)=0,\ n, k=0, 1, 2, \ldots$.

(ii) The Hamiltonian system (2.10) corresponding to $H_m$ is a completely integrable system in the Liouville sense:

$$\begin{aligned} H_m = {} & \tfrac{1}{2}\langle D\Phi, \Phi\rangle^{-2}\langle D^2\Phi, \Phi\rangle\langle D^{m+1}\Phi, \Phi\rangle - \tfrac{1}{2}\langle D\Phi, \Phi\rangle^{-1}\langle D^{m+2}\Phi, \Phi\rangle - \langle \Phi, \Psi\rangle\langle D^{m+1}\Phi, \Psi\rangle \\ & + \tfrac{1}{2}\langle \Psi, \Psi\rangle\langle D^{m+1}\Phi, \Phi\rangle + \tfrac{1}{2}\langle D^{m+1}\Psi, \Psi\rangle\langle \Phi, \Phi\rangle \\ & - \tfrac{1}{2}\sum_{j+k=m+1}\begin{vmatrix} \langle D^j\Phi, \Phi\rangle & \langle D^j\Phi, \Psi\rangle \\ \langle D^k\Phi, \Psi\rangle & \langle D^k\Psi, \Psi\rangle \end{vmatrix}. \end{aligned} \tag{2.13}$$

*Proof.* From (2.7) and (2.11), so that, $H_m = H_m^*$, i.e. $H_m$ is a real function. On the other hand, using the generating function method [8, 9] we can prove the $H_m = \Sigma_{j=1}^{2N} \lambda_j^{m+1} E_j$. The involutivitiy of $E_k$ implies the involutivity of $H_m$. From $(H_n, H_k)=0$, (ii) holds. □

## 3. The coupled HD hierarchy and nonlinearization of the Lax pairs

Now we consider the coupled HD spectral problem [7]

$$y_{xx} = (\alpha - \lambda u - \lambda^2 v)y \tag{3.1}$$

where $\lambda$ is a complex parameter, $\alpha$ is a real constant, and $u$ and $v$ are real potential functions.

We define Lenard's sequence $\{G_m,\ m=-1, 0, 1, \ldots\}$ using the following recursion relation:

$$kG_{j-1} = JG_j \qquad G_j = (b_j, b_{j+1})^T \qquad G_{-1} = (0, v^{-1/2})^T \qquad j=0, 1, 2, \ldots \tag{3.2}$$

where

$$K = \begin{pmatrix} 0 & \frac{1}{2}\partial^3 - 2\alpha\partial \\ \frac{1}{2}\partial^3 - 2\alpha\partial & u\partial + \partial u \end{pmatrix} \qquad J = \begin{pmatrix} \frac{1}{2}\partial^3 - 2\alpha\partial & 0 \\ 0 & -(v\partial + \partial v) \end{pmatrix}$$

$$\partial = \frac{\partial}{\partial x} \qquad \partial^{-1}\partial = \partial\partial^{-1} = 1. \tag{3.3}$$

From (3.2), $b_j$ $(j=0, 1, \ldots)$ are polynomials of $(u, u_x, \ldots)$ and $(v, v_x, \ldots)$. If the constant term of $G_j$ $(j=0, 1, 2)$ takes zero, $G_j$ is determined uniquely; in this case $X_m = JG_{m-1}$ is called the $m$th-order coupled HD vector field, $(u, v)_{t_m}^T = X_m$ is called the $m$th-order coupled HD equation, and $\{(u, v)_{t_m}^T = X_m,\ m=0, 1, 2, \ldots\}$ is called the coupled HD hierarchy.

*Theorem 3.1.* The $m$th-order coupled HD equation

$$(u, v)_{t_m}^T = X_m = JG_{m-1} \tag{3.4}$$

is the compatible condition for the Lax equation

$$y_{xx}=(\alpha-\lambda u-\lambda^2 v)y \tag{3.1}$$

$$y_{t_m}=\sum_{j=0}^{m}\left(-\tfrac{1}{2}b_{j-1x}\lambda^{m-j+1}y+b_{j-1}\lambda^{m-j+1}y_x\right) \tag{3.5}$$

in the case of $\lambda_{t_m}=0$, $y_{xxt_m}=y_{t_mxx}$.

*Proof.* According to (3.2) and $\lambda_{t_m}=0$, $y_{xxt_m}=y_{t_mxx}$ and by direct calculation we have (3.4). □

*Example.* The first-order coupled HD equation

$$u_{t_1}=\tfrac{1}{2}\left(\frac{1}{\sqrt{v}}\right)_{xxx}-2\alpha\left(\frac{1}{\sqrt{v}}\right)_x \qquad v_{t_1}=u_x\left(\frac{1}{\sqrt{v}}\right)+2u\left(\frac{1}{\sqrt{v}}\right)_x$$

has the Lax representation

$$y_{xx}=(\alpha-\lambda u-\lambda^2 v)y \qquad y_{t_1}=-\tfrac{1}{2}\left(\frac{1}{\sqrt{v}}\right)_x\lambda y+\frac{1}{\sqrt{v}}\lambda y.$$

The second-order coupled HD equation

$$u_{t_2}=-\tfrac{1}{4}\left[\left(\frac{1}{\sqrt{v}}\right)^3 u\right]_{xxx}+\alpha\left[\left(\frac{1}{\sqrt{v}}\right)^3 u\right]_x$$

$$v_{t_2}=\tfrac{1}{2}\left(\frac{1}{\sqrt{v}}\right)_{xxx}-2\alpha\left(\frac{1}{\sqrt{v}}\right)_x-\tfrac{3}{2}uu_x\left(\frac{1}{\sqrt{v}}\right)^3-u^2\left[\left(\frac{1}{\sqrt{v}}\right)^3\right]_x$$

has the Lax representation ($G_0=(v^{-1/2},\ -\tfrac{1}{2}uv^{-3/2})^T$)

$$y_{xx}=(\alpha-\lambda u-\lambda^2 v)y$$

$$y_{t_2}=+\tfrac{1}{4}\left[\left(\frac{1}{\sqrt{v}}\right)^3 u\right]_x\lambda y+\frac{1}{\sqrt{v}}\lambda^2 y_x-\tfrac{1}{2}\left(\frac{1}{\sqrt{v}}\right)_x\lambda^2 y-\tfrac{1}{2}u\left(\frac{1}{\sqrt{v}}\right)^3\lambda y_x.$$

*Lemma 3.1.* Let $y_j$ and $\lambda_j$ satisfy (3.1),

$$y_{jxx}=(\alpha-\lambda_j u-\lambda_j^2 v)y_j \qquad j=1,2,\ldots,N$$

then

$$K\begin{pmatrix} y_j^2 \\ \lambda_j y_j^2\end{pmatrix}=\lambda_j J\begin{pmatrix} y_j^2 \\ \lambda_j y_j^2\end{pmatrix} \qquad k\begin{pmatrix} y_j^{*2} \\ \lambda_j^* y_j^{*2}\end{pmatrix}=\lambda_j^* J\begin{pmatrix} y_j^{*2} \\ \lambda_j^* y_j^{*2}\end{pmatrix}. \tag{3.6}$$

*Proof.* Observations on the definition of $K$, $J$ and (3.1) and (3.6) are obtained through direct calculation.

From consideration of lemma 3.1 we let $\Phi_j=y_j,\ldots,\Phi_{N+j}=y_j^*$, $\Psi_j=y_{jx},\ldots,$ $\Psi_{N+j}=y_{jx}^*$, $j=1,2,\ldots,N$. Then (3.1) and its conjugate form can be written as follows:

$$\Phi_x=\Psi \qquad \Psi_x=(\alpha-Du-D^2u)\Phi. \tag{3.7}$$

Let

$$v=\langle D\Phi,\Phi\rangle^{-2} \qquad u=-2\langle D^2\Phi,\Phi\rangle\langle D\Phi,\Phi\rangle^{-3}$$

$$G_0=\begin{pmatrix}\langle D\Phi,\Phi\rangle \\ \langle D^2\Phi,\Phi\rangle\end{pmatrix}. \tag{3.8}$$

From lemma 3.1, we have

$$G_j = \begin{pmatrix} \langle D^{j+1}\Phi, \Phi\rangle \\ \langle D^{j+2}\Phi, \Phi\rangle \end{pmatrix}. \tag{3.9}$$

Since $G_j = (b_j, b_{j+1})^T$, so that $b_j = \langle D^{j+1}\Phi, \Phi\rangle$ and $b_j = b_j^*$, the time part (3.5) of the Lax pair and its conjugate form of the $m$th-order coupled HD equation (3.4) is therefore written as

$$\Phi_{t_m} = \sum_{j=0}^{m} \left(-\tfrac{1}{2}b_{j-1x}D^{m-j+1}\Phi + b_{j-1}D^{m-j+1}\Psi\right). \tag{3.10}$$

Under the condition (3.8), system (3.7) is nonlinear as follows from the Hamiltonian canonical equation

$$\Phi_x = -\frac{\partial H}{\partial \Psi} \qquad \Psi_x = \frac{\partial H}{\partial \Phi} \tag{3.11}$$

where $H = -\frac{1}{2}\langle\Psi, \Psi\rangle - \frac{1}{2}\langle D\Phi, \Phi\rangle^{-2}\langle D^2\Phi, \Phi\rangle + \frac{1}{2}\alpha\langle\Phi, \Phi\rangle$.

Under condition (3.8), the system (3.10) is nonlinear as follows from the Hamiltonian canonical equation

$$\Phi_{t_m} = -\frac{\partial H_m}{\partial \Psi} \qquad \Psi_{t_m} = \frac{\partial H_m}{\partial \Phi}$$

where $H_m$ is defined by (2.13):

$$\frac{\partial}{\partial \Phi} = \left(\frac{\partial}{\partial \Phi_1}, \dots, \frac{\partial}{\partial \Phi_{2N}}\right)^T \qquad \frac{\partial}{\partial \Psi} = \left(\frac{\partial}{\partial \Psi_1}, \dots, \frac{\partial}{\partial \Psi_{2N}}\right)^T.$$

*Theorem 3.2.* The Hamiltonian system (3.11) ($\mathbb{R}^{4N}$, $\Sigma_{j=1}^{2N}\, dP_j \wedge dQ_j$, $H$) is completely integrable in the Liouville sense.

*Proof.* From theorem 2.2, through calculation, we have $(E_k, H) = 0$, so that $(H_m, H) = 0$, $m = 1, 2, \dots, 2N$. As $H = H^*$, $H_m = H_m^*$ then (3.1) ($\mathbb{R}^{4N}, \Sigma_{j=1}^{2N}\, dP_j \wedge dQ_j$, $H$) is completely integrable in the Liouville sense.

## 4. The representation of solutions of the coupled HD equation

Consider the canonical system of the $H_m$-flow:

$$(H_m): \qquad \Phi_{t_m} = -\frac{\partial H_m}{\partial \Psi} \qquad \Psi_{t_m} = \frac{\partial H_m}{\partial \Phi}.$$

If the solution operator of its initial value problem is denoted by $g_{H_m}^{t_m}$, then its solution can be expressed as

$$\begin{pmatrix} \Phi(t_m) \\ \Psi(t_m) \end{pmatrix} = g_{H_m}^{t_m} \begin{pmatrix} \Phi_0 \\ \Psi_0 \end{pmatrix}.$$

The canonical system of the $H$-flow is as follows:

$$(H): \qquad \Phi_x = -\frac{\partial H}{\partial \Psi} \qquad \Psi_x = \frac{\partial H}{\partial \Phi}.$$

If the solution operator of its initial value problem is denoted by $g_H^x$, then its solution can be expressed as

$$\begin{pmatrix}\Phi(x)\\ \Psi(x)\end{pmatrix}=g_H^x\begin{pmatrix}\Phi_0\\ \Psi_0\end{pmatrix}.$$

Since $H_m$, $H$ are in involution, $(H_m, H)=0$, we have (see [1]):

*Proposition 4.1.*

(i) The two canonical systems $(H_m)$, $(H)$ are compatible.

(ii) The Hamiltonian phase flow $g_H^x$ and $g_{H_m}^{t_m}$ commute.

Define

$$\begin{pmatrix}\Phi(x, t_m)\\ \Psi(x, t_m)\end{pmatrix}=g_H^x g_{H_m}^{t_m}\begin{pmatrix}\Phi_0\\ \Psi_0\end{pmatrix}. \tag{4.1}$$

The commutativity of $\{g_H^x, g_{H_m}^{t_m}\}$ implies that it is a smooth function of $(x, t_m)$, which is called the involutive solution of the consistent systems of equation $(H)$, $(H_m)$.

*Theorem 4.1.* Let $(\Phi(x, t_m), \Psi(x, t_m))^T$ be an involutive solution of the consistent system $(H)$, $(H_m)$, $(u, v)^T$ and $(\Phi(x, t_m), \Psi(x, t_m))^T$ satisfy (3.8), then:

(i) the flow equations $(H)$, $(H_m)$ reduce to the spatial part (contains the conjugate part) and time part (contains the conjugate part) respectively of the Lax pair for the $m$th-order coupled HD equation with $(u, v)$ as their potential,

$$\Phi_{xx}=(\alpha-Du-D^2v)\Phi \tag{4.2}$$

$$\Phi_{t_m}=\sum_{j=0}^{m}\left(-\tfrac{1}{2}b_{j-1x}D^{m-j+1}\Phi+b_{j-1}D^{m-j+1}\Psi\right) \tag{4.3}$$

(ii) $(u, v)^T$ satisfies the $m$th-order coupled HD equation

$$\begin{pmatrix}u\\ v\end{pmatrix}_{t_m}=JG_{m-1}.$$

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